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Well-filtered algebras

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Abstract

We define a class of (lean) quasi-hereditary K-algebras A for which the standard filtration of the right regular representation may be described by a suitable directed quotient algebra A^+. For this class, projective resolutions of simple left modules over A^+ will correspond to the so-called BGG resolutions over A, defined earlier by Bernstein, Gelfand and Gelfand. In the case when K is algebraically closed and A^+ is a subalgebra of A, A^+ coincides with the concept of a Borel subalgebra of König. We show that many algebras obtained by previously defined canonical constructions belong to this class and have additional structural properties.

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1. Introduction. Well-filtered algebras

One of the key properties of quasi-hereditary algebras is the existence of a standard filtration for projective modules. This filtration is equivalent to one in which the indices of the occurring standard modules are ordered. In the case when the algebra is *lean* (cf. [1,2]), the extensions of standard modules are determined by extensions of their (simple) top factors. In this way the filtrations follow a "directed" path in the graph of the algebra A. Hence it is natural to investigate directed quotients of these algebras. Let us remark that a large portion of our results does not assume the quasi-heredity of the algebras.

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One of the authors reported the results in the Special Session on Algebraic Groups and Invariant Theory at the Winter Meeting of the American Mathematical Society in Orlando, Florida, on January 11, 1996 [4].

Let A be a basic finite dimensional algebra over a central field K. We shall fix a complete ordered set of primitive orthogonal idempotents $\boldsymbol{e}=(e_1,e_2,\dots,e_n)$ in A and refer to the algebra with such a choice as $(A,\boldsymbol{e})$. Given this order, we may also define the idempotents $\varepsilon_i=e_i+\cdots+e_n$ for $i=1,\dots,n$ and $\varepsilon_{n+1}=0$. Denote by $P(i)$, $S(i)$ and $\Delta(i)$ the indecomposable projective, simple and standard right modules, respectively. These may be identified as follows: $P(i)=e_iA$, $S(i)=e_iA/e_i\operatorname{rad}A$ and $\Delta(i)=e_iA/e_iA\varepsilon_{i+1}A$. Thus, denoting by $V(i)$ the module $e_iA\varepsilon_{i+1}A$, we get the following canonical short exact sequences for $1\le i\le n$:

$$0\to V(i)\to P(i)\to \Delta(i)\to 0.$$

The corresponding left modules will be denoted by $P^\circ(i)$, $S^\circ(i)$, $\Delta^\circ(i)$ and $V^\circ(i)$.

Let us also recall that the algebra $(A,\boldsymbol{e})$ is called *quasi-hereditary* if the standard modules are Schurian (i.e. their endomorphism rings are division rings) and the regular representation A_A has a filtration with factors isomorphic to standard modules $\Delta(i)$.

For further notations and definitions, as well as for some basic results and background we refer to [1, 7, 9, 11, 12]

Definition 1.1. Let $(A,\boldsymbol{e})$ be a finite dimensional algebra with a given order $\boldsymbol{e}=(e_1, e_2,\dots,e_n)$ of a complete set of primitive orthogonal idempotents. Define the ideal $I^+=I^+(A,\boldsymbol{e})$ of A to be the ideal generated by the sets e_jAe_i for $1\le i<j\le n$. Thus $I^+=\sum_{j>i}Ae_jAe_iA$. Similarly, we may define the ideal $I^-=\sum_{i<j}Ae_iAe_jA$.

Definition 1.2. For a given algebra $(A,\boldsymbol{e})$ let $A^+=(A,\boldsymbol{e})^+$ be the quotient of A modulo the ideal I^+ defined above. Similarly, $A^-=A/I^-$.

Thus the algebras A^+ and A^- are the maximal directed quotients of A with respect to the given order $\boldsymbol{e}$. Clearly, if $(A,\boldsymbol{e})$ is a quasi-hereditary algebra, both $(A^+,\boldsymbol{e})$ and $(A^-,\boldsymbol{e})$ are quasi-hereditary and $(A^+)^+=A^+$, $(A^-)^-=A^-$, while $(A^+)^-\simeq(A^-)^+\simeq A/\operatorname{rad}A$.

Let us observe that $A^-\simeq((A^{op})^+)^{op}$. Thus, although many of our statements in terms of A^+ will have their dual counterparts concerning A^-, we shall usually refrain from formulating them explicitly.

It is clear from the definition that I^+ is the ideal generated by the submodules $V^\circ(i)$ of the left regular representation ${}_AA$, i.e. $I^+=\sum_{i=1}^n V^\circ(i)A$. If A is quasi-hereditary, then the multiplication maps $A\varepsilon_i\otimes_{\varepsilon_iA\varepsilon_i}\varepsilon_iA\to A\varepsilon_iA$ are bijective for every $1\le i\le n$ [12] and thus the standard filtration of A_A is controlled by the left standard modules $\Delta^\circ(i)\simeq P^\circ(i)/V^\circ(i)$. Therefore it is natural to consider the case when the ideal I^+ coincides with the sum of submodules $V^\circ(i)$ and, as a consequence, A^+ defines the standard filtration of A_A.

Definition 1.3. We call an algebra $(A, \mathbf{e})$ *right well-filtered* if $\bigoplus_{i=1}^{n} V^\circ(i)$ is an ideal in A. One may similarly define *left well-filtered* algebras.

The next three propositions list several characterizations of this property.

Proposition 1.4. *Let A be an algebra with a given order $\mathbf{e}$ of the primitive idempotents. Then the following conditions are equivalent*:

(i) *A is right well-filtered*;

(ii) *$V^\circ(i)Ae_k \subseteq V^\circ(k)$ for every index i,k (equivalently, for every index $i<k$)*;

(ii)′ *$\varepsilon_{i+1}Ae_iAe_k \subseteq A\varepsilon_{k+1}Ae_k$ for every index i,k (equivalently, for every index $i<k$)*;

(ii)″ *$e_jAe_iAe_k \subseteq e_jA\varepsilon_{k+1}Ae_k$ for every index k and every index $i<j$ (equivalently, for every index $i<j\le k$)*;

(iii) *$\varphi(V^\circ(i)) \subseteq V^\circ(k)$ for every index i,k (equivalently, for every index $i<k$) and for every homomorphism $\varphi : P^\circ(i) \to P^\circ(k)$*;

(iv) *$I^+ = \bigoplus_{i=1}^{n} V^\circ(i)$.*

Proof. (ii) and (iv) are simple reformulations of the definition of a right well-filtered algebra, i.e. that $\bigoplus_i V^\circ(i)$ is an ideal of A. The conditions (ii)′ and (ii)″ are reformulations of (ii) in terms of the idempotents. Finally, the equivalence of (ii) and (iii) follows from the fact that homomorphisms $P^\circ(i) \to P^\circ(k)$ are precisely the right multiplications by elements of Ae_k. □

The next set of conditions characterizes right well-filtered algebras in terms of properties of the left standard modules $\Delta^\circ(i)$.

Proposition 1.5. *Let A be an algebra with a given order $\mathbf{e}$ of the primitive idempotents. Then the following conditions are equivalent*:

(i) *A is right well-filtered*;

(ii) *$\varepsilon_{i+1}Ae_i\Delta^\circ(k) = 0$ for every index i,k (equivalently, for every index $i<k$)*;

(iii) *the natural homomorphisms $P^\circ(i) \to \Delta^\circ(i)$ will induce isomorphisms $\mathrm{Hom}(\Delta^\circ(i), \Delta^\circ(k)) \simeq \mathrm{Hom}(P^\circ(i), \Delta^\circ(k))$ for every index i,k (equivalently, for every index $i<k$)*;

(iii)′ *$\mathrm{Hom}(V^\circ(i), \Delta^\circ(k)) \simeq \mathrm{Ext}^1_A(\Delta^\circ(i), \Delta^\circ(k))$ for every index i,k (equivalently, for every index $i<k$)*;

(iv) *${}_AA^+ \simeq \bigoplus_{i=1}^{n} \Delta^\circ(i)$.*

Proof. The equivalence of the conditions (i)–(iv) can be proved using the parallel conditions of Proposition 1.4. Condition (ii) is a simple reformulation of Proposition 1.4 (ii)′. The equivalence of (iii), and Proposition 1.4 (iii) follows from the fact that any homomorphism $P^\circ(i) \xrightarrow{\varphi} \Delta^\circ(k)$ gives rise to a homomorphism $P^\circ(i) \xrightarrow{\bar{\varphi}} P^\circ(k)$. The equivalence of (iii) and (iii)′ can be obtained from the long exact sequence

$$0 \to \mathrm{Hom}(\Delta^\circ(i), \Delta^\circ(k)) \to \mathrm{Hom}(P^\circ(i), \Delta^\circ(k)) \to \mathrm{Hom}(V^\circ(i), \Delta^\circ(k))$$
$$\to \mathrm{Ext}^1_A(\Delta^\circ(i), \Delta^\circ(k)) \to \mathrm{Ext}^1_A(P^\circ(i), \Delta^\circ(k)) \to \cdots.$$

Finally, the equivalence of (iv) and Proposition 1.4 (iv) is straightforward. □

Let us now recall that for a given algebra $(A, \mathbf{e})$ the *trace filtration* of a module X_A is given by the sequence $0 \subseteq X\varepsilon_n A \subseteq X\varepsilon_{n-1}A \subseteq \cdots \subseteq X\varepsilon_1 A = X$: the consecutive terms are the traces (i.e. sums of all homomorphic images) of the projective modules $\varepsilon_i A = P(i) \oplus P(i+1) \oplus \cdots \oplus P(n)$ on X. One may similarly define the *reverse trace filtration* of X by taking the sequence $0 \subseteq Xe_1 A \subseteq X(e_1 + e_2)A \subseteq \cdots \subseteq X(e_1 + e_2 + \cdots + e_n)A = X$, i.e. the trace filtration of X with respect to the opposite order.

Finally, let us recall that a BGG resolution of a module M is an exact sequence $0 \to X_k \to X_{k-1} \to \cdots \to X_0 \to M \to 0$ such that each term X_i is a direct sum of standard modules (cf. [6]).

Proposition 1.6. *Let A be an algebra with a given order $\mathbf{e}$ of the primitive idempotents. Then the following conditions are equivalent:*

(i) *A is right well-filtered;*

(ii) *for each index k, every factor of the reverse trace filtration of $\Delta^\circ(k)$ is a homogeneous module (i.e. its composition factors are all isomorphic);*

(iii) *for each index k, $\Delta^\circ(k)$ has a composition series where the sequence of the indices of composition factors is monotone;*

(iv) *for each index k, the trace of $\bigoplus_{j \le k} \Delta^\circ(j)$ in* rad $\Delta^\circ(k)$ *is the entire* rad $\Delta^\circ(k)$;

(v) *every simple left module $S^\circ(i)$ has a BGG resolution;*

(vi) *every left A^+-module as a left A-module has a BGG resolution.*

Proof. (i)⇒(ii): Consider the reverse trace filtration

$$0 \subseteq Ae_1\Delta^\circ(k) \subseteq A(e_1 + e_2)\Delta^\circ(k) \subseteq \cdots \subseteq A(e_1 + \cdots + e_k)\Delta^\circ(k) = \Delta^\circ(k).$$

If X_i is the factor $A(e_1 + \cdots + e_i)\Delta^\circ(k)/A(e_1 + \cdots + e_{i-1})\Delta^\circ(k)$, then $\varepsilon_{i+1}X_i = 0$ follows from the condition (ii) of Proposition 1.5, while $(e_1 + \cdots + e_{i-1})X_i = 0$ is obvious, hence the composition factors of X_i are all isomorphic to $S^\circ(i)$.

(ii)⇒(iii): Let us consider a refinement of the reverse trace filtration into a composition series. Then the sequence of indices of the composition factors is clearly monotone.

(iii)⇒(iv): Note that if a module X has a composition series where the sequence of indices of composition factors is monotone (the smallest index corresponding to a simple submodule in the socle of X) then the submodules of X also have such composition series. Furthermore, if the image of a map $P^\circ(j) \xrightarrow{f} X$ possesses such a composition series then $V^\circ(j) \subseteq \mathrm{Ker}\, f$ must hold. Hence, assuming (iii), the obvious fact that rad $\Delta^\circ(k)$ is generated by homomorphic images of the projective modules $P(j)$, $j \le k$, implies (iv).

(iv) ⇒ (i): We shall prove that under the assumption, if a module X is a homomorphic image of a left standard module $\Delta^\circ(k)$ for some k then $\varepsilon_{i+1}Ae_iX = 0$ for every $1 \le i \le n$. In view of the condition (ii) of Proposition 1.5, this will prove our statement. We shall use induction on the Loewy length of X. The statement is clearly true if X is simple, furthermore it is also true for arbitrary indices i and k, whenever $i \ge k$. Thus, assume now that there exists an epimorphism $\Delta^\circ(k) \to X$ and let $i < k$. Then clearly $\varepsilon_{i+1}Ae_iX = \varepsilon_{i+1}Ae_i \operatorname{rad} X$. Here $\operatorname{rad} X$ is a homomorphic image of $\operatorname{rad} \Delta^\circ(k)$ which is, by assumption, generated by homomorphic images of $\Delta^\circ(j)$, $j \le k$. Hence, $\operatorname{rad} X$ is a sum of submodules X_i which have smaller Loewy length than X and which are homomorphic images of left standard modules. Thus, we may apply the induction hypotheses to get the statement for X.

(i) ⇒ (vi): Condition (iv) of Proposition 1.5 implies that the projective resolution of any left A^+-module gives a BGG resolution over A.

Since the implication (vi) ⇒ (v) is obvious, it is enough to show that (v) ⇒ (iv), but this is also clear from the definition of a BGG resolution. □

The concept of a right well-filtered algebra yields immediately the concept of one sided lean algebras. Recall that an algebra $(A, \boldsymbol{e})$ is *lean* with respect to the order $\boldsymbol{e}$ (see [1]) if $e_j \operatorname{rad}^2 Ae_k \subseteq e_j \operatorname{rad} A\varepsilon_m \operatorname{rad} Ae_k$ for every j, k, where $m = \min\{j, k\}$. We call the algebra $(A, \boldsymbol{e})$ *right lean* if the above condition holds whenever $j \le k$. Note that for algebras with Schurian standard modules this is equivalent to the fact that $V(i)$ is a *top submodule* of $\operatorname{rad} P(i)$ for every index i. Further homological characterizations can also be derived from [2].

Corollary 1.7. *If $(A, \boldsymbol{e})$ is right well-filtered then $(A, \boldsymbol{e})$ is right lean.*

Proof. Note that $e_iAe_j = e_i \operatorname{rad} Ae_j$ for $i \ne j$, hence using the condition (ii)″ of Proposition 1.4, we have, for $j \le k$, $e_j \operatorname{rad}^2 Ae_k \subseteq e_j \operatorname{rad} A(e_1 + \cdots + e_{j-1}) \operatorname{rad} Ae_k + e_j \operatorname{rad} A\varepsilon_j \operatorname{rad} Ae_k \subseteq e_jA\varepsilon_{k+1}Ae_k + e_j \operatorname{rad} A\varepsilon_j \operatorname{rad} Ae_k \subseteq e_j \operatorname{rad} A\varepsilon_j \operatorname{rad} Ae_k$. □

In what follows we shall be interested mostly in cases where the standard modules are Schurian. Thus it is worth mentioning that for a right well-filtered algebra $(A, \boldsymbol{e})$, the condition that the standard modules of $(A, \boldsymbol{e})$ are Schurian (in other words, $e_i \operatorname{rad} Ae_i \subseteq e_iA\varepsilon_{i+1}Ae_i$ for every $1 \le i \le n$) is equivalent to the fact that the species of A has no loops, i.e. that $e_i \operatorname{rad} Ae_i \subseteq e_i \operatorname{rad}^2 Ae_i$ for every $1 \le i \le n$. The Schurian condition obviously implies that the species has no loops, and the other direction of the equivalence follows from the condition (ii)″ of Proposition 1.4.

Example 1.8. The following example shows that the concept of a right well-filtered algebra is indeed one-sided. The example also shows that the converse of Corollary 1.7 does not hold. Let

$$A_A = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 \quad \text{and} \quad {}_AA = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}$$

be the right and left regular representations of the path algebra A over a field K, respectively. The algebra A is clearly lean but it is not right well-filtered. Indeed, $\bigoplus_i V^\circ(i) = V^\circ(1)$ is not an ideal, because there is a map $P^\circ(1) \to P^\circ(4)$, sending $V^\circ(1)$ non-trivially into $P^\circ(4) = \Delta^\circ(4)$. Note that A is left well-filtered.

Some special classes of quasi-hereditary algebras are well-filtered. (We refer to [1] or [12] for the definition of shallow and replete quasi-hereditary algebras.)

Proposition 1.9. *Assume $(A, \mathbf{e})$ is a shallow quasi-hereditary algebra. Then $(A, \mathbf{e})$ is both right and left well-filtered and* $\mathrm{rad}^2 A^+ = 0$.

Proof. Since the standard and costandard modules of shallow algebras have semisimple radicals, the first statement follows from the condition (iv) of Proposition 1.6, while the second from the condition (iv) of Proposition 1.5. □

On the other hand, Example 1.8 shows that a replete algebra is in general not well-filtered. Note, that in this example the replete algebra A is not well-filtered because of commutativity relations in the definition of A. The canonical replete algebras, which are necessarily monomial are always well-filtered, due to the following proposition.

Proposition 1.10. *Let $A = K\Gamma/I$ be a monomial algebra (i.e. a path algebra over the graph Γ modulo relations which are generated by paths). Then A is right lean if and only if A is right well-filtered.*

Proof. In view of Corollary 1.7 we have to show only that if A is right lean then A is right well-filtered. However, this follows from the fact that for right lean monomial algebras $e_jAe_iAe_k = 0$ for $i < j \le k$. Hence A is well-filtered by (ii)″ of Proposition 1.4. □

Next, we examine the behaviour of the construction of A^+ and the well-filtered property in connection with some other standard constructions. For each algebra $(A, \mathbf{e})$ we may define two sequences of algebras: $B_t(A) = A/A\varepsilon_{t+1}A$ and $C_t(A) = \varepsilon_t A \varepsilon_t$ (cf. [12]).

Proposition 1.11. *Let $(A, \mathbf{e})$ be given.*
(a) *If A is right well-filtered, then so are the algebras $B_t(A)$ for $1 \le t \le n$.*
(b) *If A is right well-filtered, then so are the algebras $C_t(A)$ for $1 \le t \le n$.*

Proof. Both statements follow immediately from Proposition 1.4(ii)′. □

Observe that although $B_t(A)^+ \simeq B_t(A^+)$ holds for any algebra A and for all $1 \le t \le n$, the algebras $C_t(A)^+$ and $C_t(A^+)$ are, in general, not isomorphic. In fact, this leads to yet another characterization of well-filtered algebras.

Proposition 1.12. *(A,e) is right well-filtered if and only if $C_t(A)^+ \simeq C_t(A^+)$ for every $1 \le t \le n$.*

Proof. Let us use the notation $I_t^+ = \sum_{\ell \ge t} V^\circ(\ell)A = \sum_{\ell \ge t} A\varepsilon_{\ell+1}Ae_\ell A$. We show first that for a given t, $C_t(A)^+ \simeq C_t(A^+)$ if and only if

$$e_jAe_iAe_k \subseteq I_t^+ \text{ for every } i < t \le j \le k.$$

Indeed, since $C_t(A)^+ = \varepsilon_t A\varepsilon_t/\varepsilon_t I_t^+ \varepsilon_t$ and $C_t(A^+) = \varepsilon_t A \varepsilon_t/\varepsilon_t I^+ \varepsilon_t$, the two algebras are isomorphic if and only if $\varepsilon_t I^+ \varepsilon_t \subseteq I_t^+$ for every t. Using the definition of I^+, this condition is equivalent to the inclusion $e_jA\varepsilon_{i+1}Ae_iAe_k \subseteq I_t^+$ for every $i < t \le j,k$. Since $e_jAe_iAe_k \subseteq e_jA\varepsilon_jAe_iAe_k \subseteq e_jA\varepsilon_{i+1}Ae_iAe_k$, the left side of the inclusion can be simplified to $e_jAe_iAe_k$, and we can make the restriction $j \le k$ for the indices, since for $k < j$ the inclusion always holds. This gives us the desired formula.

It follows from the above formula that $C_t(A)^+ \simeq C_t(A^+)$ for every t if and only if

$$e_jAe_iAe_k \subseteq I_j^+ \quad \text{for every } i < j \le k.$$

Suppose first, that A is right well-filtered. Then the condition (ii)$''$ of Proposition 1.4 gives that $e_jAe_iAe_k \subseteq V^\circ(k) \subseteq I_j^+$ for every $i < j \le k$.

Assume now that $e_jAe_iAe_k \subseteq I_j^+$ for every $i < j \le k$. We shall prove by reverse induction on j (with fixed i and k) that $e_jAe_iAe_k \subseteq V^\circ(k)$. For $j = k$ we have $e_jAe_iAe_k \subseteq I_j^+e_k = I_k^+e_k \subseteq V^\circ(k)$. Now suppose that we have proved the statement for every j' with $j < j' \le k$. Then $e_jAe_iAe_k \subseteq I_j^+e_k = \sum_{\ell \ge j} V^\circ(\ell)Ae_k = \sum_{\ell \ge j} A\varepsilon_{\ell+1}Ae_\ell Ae_k \subseteq V^\circ(k)$ by the induction hypothesis. Thus A must be right well-filtered by Proposition 1.4(ii)$''$. □

Many homological aspects of the algebra A are encoded in the so-called *Ext-algebra* A^* of A: $A^* = \bigoplus_{k \ge 0} \operatorname{Ext}_A^k(A/\operatorname{rad} A, A/\operatorname{rad} A)$ (see, e.g., [3]). Note that one can use the identity maps of the corresponding simple modules to get a complete set of primitive orthogonal idempotents f_i, $1 \le i \le n$, in A^*. If $\boldsymbol{e} = (e_1, e_2, \dots, e_n)$ defines the order of the idempotents in A then the natural order of the idempotents f_i in A^* is the *reverse* order $\boldsymbol{f} = (f_n, f_{n-1}, \dots, f_1)$. Hence $(A^*)^+$ is defined with respect to this order $\boldsymbol{f}$.

In the case of monomial algebras A, there is a combinatorial description of A^* (see [13] or [3]). Namely, if $A = K\Gamma/I$ is monomial, then A^* can be identified with a K-algebra, given by a multiplicative basis $\tilde{\Gamma}$, where $\tilde{\Gamma}$ consists of all vertices (identified with the idempotent elements $e_1, e_2, \dots, e_n$ in A) and arrows of the quiver Γ of A, as well as all paths p in Γ which can be written as the concatenation of subpaths $p = p_1p_2 \dots p_t$, where p_1 is an arrow of Γ, none of the subpaths p_i is 0 in A (i.e. $p_i \notin I$), and p_ip_{i+1} is a right-minimal 0-path in A. (Note that a path p is called *right minimal* 0*-path* if $p \in I$ and there are no subpaths p' and p'' of p of non-zero length so that $p = p' \cdot p''$ and $p' \in I$.) The product $p \cdot p'$ of two basis elements p and p' is defined to be the concatenation $p'p$ provided $p'p \in \tilde{\Gamma}$ and 0 otherwise.

We have the following statement for monomial algebras.

Proposition 1.13. *Let $A = K\Gamma/I$ be a lean quasi-hereditary monomial algebra. Then* $(A^*, \boldsymbol{f})^+ \simeq ((A, \boldsymbol{e})^+)^*$.

Proof. To simplify the notation, we shall refer to the algebras above as $(A^*)^+$ and $(A^+)^*$.

Note that A^+ is also monomial and hence for the K-basis $\tilde{\Gamma}^+$ of $(A^+)^*$ we can choose all those paths from the basis $\tilde{\Gamma}$ of A^* (including paths of length 0 and 1), for which the sequence of vertices along the path is monotone non-decreasing, according to the order given by $\boldsymbol{e}$. Thus, we may assume that $(A^+)^* \leq A^*$ in a canonical way.

Now, observe that $I^+(A^*, \boldsymbol{f})$ is generated, as a K-subspace of A^*, by elements $p \cdot f_i \cdot q \cdot f_j \cdot r = re_j q e_i p \in \tilde{\Gamma}$ (where $p, q, r \in \tilde{\Gamma}$ and $i < j$), so $I^+(A^*, \boldsymbol{f})$ is included in the subspace generated by $\tilde{\Gamma} \setminus \tilde{\Gamma}^+$. Thus to prove the required isomorphism $(A^+)^* \simeq (A^*)^+$, it is sufficient to show that $\tilde{\Gamma} \setminus \tilde{\Gamma}^+ \subseteq I^+(A^*, \boldsymbol{f})$.

So, let $p \in \tilde{\Gamma} \setminus \tilde{\Gamma}^+$. Without loss of generality we may assume that p cannot be written as a product of two non-idempotent elements of $\tilde{\Gamma}$ in A^*. Since A is lean, and since Γ does not contain loops (by the quasi-heredity of A), Lemma 5.1(iv) of [3] implies that p does not contain a subpath of length 2 whose middle vertex is minimal (with respect to the order $\boldsymbol{e}$). Thus, either p is monotone or p is the concatenation $p'p''$ of paths p' and p'' such that p' is increasing and p'' is decreasing. It is enough to show that the latter is impossible.

Let $p = p_1 p_2 \dots p_t$ be the "canonical decomposition" of p (described above). Suppose that k is minimal such that p_k and p'' have at least one arrow in common. Then consider the path $p_{k-1} p_k = q'q''$ where q' and q'' are subpaths of p' and p'', respectively. Here q'' is a non-zero path because it is a subpath of p_k; q' is a non-zero path because $p_{k-1} p_k$ is a right-minimal 0-path; while $q'q''$ is a 0-path. Taking into account that q' is increasing and q'' is decreasing, this contradicts the fact that A is quasi-hereditary (cf. Lemma 5.3(ii) of [3]). □

It is easy to construct examples showing that, in general, the previous statement is not true without the assumption on A to be lean.

Example 1.14. The following path algebra A is monomial but not lean:

$$A_A = \begin{smallmatrix} 1 \\ 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 .$$

Then A^* and $(A^*)^+$ are given by

$$_{A^*}A^* = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 \quad \text{and} \quad _{(A^*)^+}(A^*)^+ = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 .$$

On the other hand the regular representations of the algebras A^+ and $(A^+)^*$ are

$$A^+{}_{A^+} = \begin{smallmatrix} 1 \\ 3 \\ 4 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 \quad \text{and} \quad _{(A^+)^*}(A^+)^* = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 .$$

Hence $(A^+)^* \not\simeq (A^*)^+$.

Example 1.15. Consider the lean algebra A of Example 1.8. Here the regular representations of A^*, $(A^*)^+$, A^+ and $(A^+)^*$ are

$$ {}_{A^*}A^* = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4, \qquad {}_{(A^*)^+}(A^*)^+ = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4, $$

$$ A^+{}_{A^+} = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 \quad \text{and} \quad {}_{(A^+)^*}(A^+)^* = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4. $$

Hence, again, $(A^+)^* \not\simeq (A^*)^+$.

Let us remark that, although the general statement about the isomorphism of $(A^+)^*$ and $(A^*)^+$ is not true, the isomorphism holds for some other classes of algebras, as we will show in Section 2.

Regarding the well-filtered property, we have the following statement.

Proposition 1.16. *Let A be a monomial algebra, which is quasi-hereditary. If $(A, \mathbf{e})$ is both right and left well-filtered, then $(A^*, \mathbf{f})$ is also both right and left well-filtered.*

Proof. By Proposition 1.10, A is lean, and by Corollary 5.6 of [3], A^* is a quasi-hereditary lean algebra. Since A^* is not necessarily monomial, this would not automatically imply that A^* is well-filtered, however from the proof of Corollary 5.6 in [3] one can conclude that the following implications hold in A^*: if $f_j \operatorname{rad} A^* f_i \operatorname{rad} A^* f_k \neq 0$, then $i < \max\{j,k\}$. Hence A^* is both right and left well-filtered with respect to the order given by $\mathbf{f}$. □

Simple examples of non-monomial algebras show that, in general, Proposition 1.16 does not hold.

Example 1.17. Consider the following path algebra:

$$ A_A = \begin{smallmatrix} & 1 & \\ 2 & & 4 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 5 & \\ 4 & & 6 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 3 \end{smallmatrix}. $$

The algebra A is quasi-hereditary and shallow, and therefore both right and left well-filtered. On the other hand, the left structure of the Ext-algebra A^* can be described by the same Loewy diagram and thus A^* is neither right nor left well-filtered (with respect to the opposite order).

2. Standard filtrations and the pushdown functor $\mathbf{F}^+$

Given a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{\ell-1} \subseteq M_\ell = M$, we will occasionally specify the embeddings $\iota_t : M_t \to M_{t+1}$ for $0 \leq t \leq \ell - 1$.

A defining feature of quasi-hereditary algebras is that they possess a *standard filtration* (*Δ-filtration*), i.e. A_A has a filtration where the factors of the filtration are isomorphic to one of the standard modules $\Delta(i)$. The class of A-modules having a standard filtration will be denoted by $\mathscr{F}(\Delta)$. Our next goal is to examine the effect of the "pushdown" functor $F^+ : \text{mod-}A \to \text{mod-}A^+$ on these filtrations. Here the functor F^+ is defined by $F^+(M) = M \otimes_A A^+ \simeq M/MI^+$, hence F^+ is right exact.

Thus, from now on we shall assume that A is quasi-hereditary.

Let $G : \text{mod-}A \to \text{mod-}B$ be an (additive) right exact functor. We will say that the functor G *preserves the filtration* $0 = M_0 \xrightarrow{\iota_0} M_1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{\ell-1}} M_\ell = M$ of a module M if $G(\iota_i)$ is an embedding for $0 \le i \le \ell - 1$, that is, if the sequence $0 = G(M_0) \xrightarrow{G(\iota_0)} G(M_1) \xrightarrow{G(\iota_1)} \cdots \xrightarrow{G(\iota_{\ell-1})} G(M_\ell) = G(M)$ gives a filtration of the module $G(M)$. Note that the right exactness of G implies that the factors of the filtration of $G(M)$ will be the G-images of the factors of the original filtration of M.

It is easy to see that the functor F^+ maps the standard modules $\Delta(i)$ into the simple modules $S(i)$, considered as A^+-modules. Thus if F^+ preserves the standard filtration of a module M, then the image of this filtration will give a composition series of $F^+(M)$.

Lemma 2.1. *Let $M \in \mathscr{F}(\Delta)$ be a module with a standard filtration. Then F^+ preserves this filtration of M if and only if the composition length $\ell(F^+(M))$ of the module $F^+(M)$ equals the length $\ell_\Delta(M)$ of the standard filtration of M.*

Proof. By induction on the length of the filtration of M. □

Corollary 2.2. *If F^+ preserves one standard filtration of a module M then it preserves every standard filtration of M.*

Thus we can speak about *well-filtered modules*: these are those modules from $\mathscr{F}(\Delta)$ for which the functor F^+ preserves the standard filtration. Let us denote the class of well-filtered modules by $\mathscr{W}\mathscr{F}(\Delta)$. It is easy to see that $\mathscr{W}\mathscr{F}(\Delta)$ is closed under taking direct sums and direct summands.

Theorem 2.3. *Let (A, e) be a quasi-hereditary algebra. Then the following conditions are equivalent:*

(i) *A is right well-filtered.*

(ii) *F^+ preserves the standard filtration of A_A, i.e., $A_A \in \mathscr{W}\mathscr{F}(\Delta)$.*

(iii) *F^+ preserves the standard filtration of any module $M \in \mathscr{F}(\Delta)$, that is, $\mathscr{F}(\Delta) = \mathscr{W}\mathscr{F}(\Delta)$.*

(iv) *The restriction of the functor F^+ to the category $\mathscr{F}(\Delta)$ is exact.*

Proof. First let us introduce some notation. Let d_i denote the K-dimension of the simple module $S(i)$. Clearly, $d_i = \dim_K S(i) = \dim_K S^\circ(i)$. For a module $M \in \mathscr{F}(\Delta)$ we shall denote by $[M : \Delta(i)]$ the number of factors in a standard filtration of M which are isomorphic to $\Delta(i)$. It is easy to show that this number is well-defined, i.e. it is independent of the choice of the particular filtration of M. Clearly, the length of the standard filtration of M can be obtained as $\ell_\Delta(M) = \sum_i [M : \Delta(i)]$. Similarly, for a module M we shall denote by $[M:S(i)]$ the number of composition factors of M, isomorphic to $S(i)$. It is easy to see that $[M:S(i)]=\frac{1}{d_i}\dim_K Me_i$.

(i) ⇒ (ii). If A is well-filtered, then, by the condition (iv) of Proposition 1.5, ${}_AA^+ \simeq \bigoplus_{i=1}^n \Delta^\circ(i)$. Using the Bernstein–Gelfand–Gelfand Reciprocity Principle (cf. [7, 12]), we get the following:

$$\begin{aligned}\ell_\Delta(A) &= \sum_j [A_A:\Delta(j)] = \sum_{i,j}[P(i):\Delta(j)] = \sum_{i,j}\frac{d_i}{d_j}[\Delta^\circ(j):S^\circ(i)]\\ &= \sum_{i,j}\frac{d_i}{d_j}\cdot\frac{1}{d_i}\dim_K e_i\Delta^\circ(j) = \sum_{i,j}\frac{1}{d_j}\dim_K e_iA^+e_j\\ &= \sum_{i,j}[e_iA^+:S(j)] = \ell(A^+).\end{aligned}$$

Hence, F^+ preserves the standard filtration of A_A. Note that this numerical argument can be reversed to show that if (ii) holds, then ${}_AA^+$ cannot be a proper quotient of $\bigoplus_{i=1}^n \Delta^\circ(i)$. Thus Proposition 1.5 implies that A is well-filtered, i.e. (ii) ⇒ (i) holds.

To show (ii) ⇒ (iii), let $0\to X\to Y\to Z\to 0$ be a short exact sequence of modules in $\mathscr{F}(\Delta)$. If $Y\in\mathscr{WF}(\Delta)$, then $X,Z\in\mathscr{WF}(\Delta)$. Indeed, F^+ maps, on the one hand, the short exact sequence given above into an exact sequence $F^+(X)\to F^+(Y)\to F^+(Z)\to 0$, hence $\ell(F^+(Y))\le\ell(F^+(X))+\ell(F^+(Z))$. On the other hand, by assumption, $\ell(F^+(Y)) =\ell_\Delta(Y)$, and clearly $\ell_\Delta(Y)=\ell_\Delta(X)+\ell_\Delta(Z)\ge\ell(F^+(X))+\ell(F^+(Y))$. Hence all the inequalities must be equalities, and thus $F^+(X)$ and $F^+(Z)$ belong to $\mathscr{WF}(\Delta)$. Note also that in this case (i.e. when $Y\in\mathscr{WF}(\Delta)$), the map $F^+(X)\to F^+(Y)$ must be a monomorphism.

Since $\mathscr{WF}(\Delta)$ is closed under taking direct sums, (ii) implies that every free module F is well-filtered, i.e. $F\in\mathscr{WF}(\Delta)$. Now, recall that $\mathscr{F}(\Delta)$ is closed under taking kernels of epimorphisms (cf. [11] or [12]), hence for any module $M\in\mathscr{F}(\Delta)$ there is a short exact sequence $0\to X\to F\to M\to 0$ for some free module F and $X,F\in\mathscr{F}(\Delta)$. Thus, by the previous considerations we get that $M\in\mathscr{WF}(\Delta)$, proving the implication (ii) ⇒ (iii).

Note that we have proved that F^+ is exact on those short exact sequences in $\mathscr{F}(\Delta)$ in which the middle term is well-filtered. Hence (iii) ⇒ (iv).

Finally, the implication (iv) ⇒ (iii) follows from the definition of $\mathscr{WF}(\Delta)$, while the implication (iii) ⇒ (ii) is trivial. □

Corollary 2.4. *Let A be a quasi-hereditary algebra which is right well-filtered. Then $proj.dim_{A^+} S(i) \le proj.dim_A \Delta(i)$ for every index i. In particular, $gl.dim\, A^+ \le gl.dim\, A$.*

Proof. Take a projective resolution of the standard A-module $\Delta(i)$. Since $\mathscr{F}(\Delta)$ is closed under taking kernels of epimorphisms, this long exact sequence is a product of short exact sequences in $\mathscr{F}(\Delta)$. By the exactness of F^+ on the subcategory $\mathscr{F}(\Delta)$, we get a projective resolution of $F^+(\Delta(i))=S_{A^+}(i)$. Hence the statements of Corollary 2.4 follow. □

The following example shows that if the algebra A is not well-filtered, the above statements do not hold in general.

Example 2.5. Consider the following path algebra:

$$A_A = \begin{smallmatrix} 1 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5.$$

Here we have:

$$A^+_{A^+} = \begin{smallmatrix} 1 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5.$$

The algebra A is not right well-filtered, $gl.dim A = 2$ and $gl.dim A^+ = 3$.

Corollary 2.4 immediately implies the first part of the following statement.

Proposition 2.6. *Let A be a replete quasi-hereditary algebra which is right well-filtered. Then A^+ is hereditary, furthermore $(A^+)^* \simeq (A^*)^+$.*

Proof. Since A is right well-filtered, Corollary 2.4 yields $proj.dim_{A^+} S(i) \leq proj.dim_A \Delta(i)$. Since A is replete, $proj.dim_A \Delta(i) \leq 1$ for $1 \leq i \leq n$. Thus A^+ is hereditary.

In particular, this implies that $(A^+)^* = \bigoplus_{k=0}^{1} \bigoplus_{i,j} \mathrm{Ext}^k_{A^+}(S(i),S(j))$ as a vector space. One can see easily that $\mathrm{Hom}_A(S(i),S(j)) \simeq \mathrm{Hom}_{A^+}(S(i),S(j))$, and

$$\mathrm{Ext}^1_{A^+}(S(i),S(j)) \simeq \begin{cases} \mathrm{Ext}^1_A(S(i),S(j)) & \text{for } i<j; \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $A^* = \bigoplus_k \bigoplus_{i,j} \mathrm{Ext}^k_A(S(i),S(j))$, and since $(A,\boldsymbol{e})$ is replete, Corollary 4.6 of [3] implies that $(A^*,\boldsymbol{f})$ is shallow. (Recall that the quasi-heredity of A^* and the definition of $(A^*)^+$ relate to the (reverse) order $\boldsymbol{f}$.) It also follows from [3] that $\mathrm{rad}^2 A^* = \bigoplus_{k \geq 2} \bigoplus_{i,j} \mathrm{Ext}^k_A(S(i),S(j))$. Proposition 1.9 implies that $\mathrm{rad}^2 (A^*)^+ = 0$, hence $(A^*)^+ \simeq (A^*/\mathrm{rad}^2 A^*)^+ = \bigoplus_{k=0}^{1} \bigoplus_{i>j} \mathrm{Ext}^k_A(S(i),S(j))$.

Since the multiplication structure is clearly the same in both cases, we get the required isomorphism. □

Theorem 2.3 shows that for a well-filtered quasi-hereditary algebra A, every standard filtration of A_A yields a composition series of A^+ with factors given by the tops of the corresponding standard modules in the given standard filtration of A_A. The following example shows that in general, not every composition series will arise in this way.

Example 2.7. Consider the path algebra given by

$$A_A = \begin{smallmatrix} & 1 & \\ 2 & & 4 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}.$$

Then the regular representation of A^+ is as follows:

$$A^+_{A^+} = \begin{smallmatrix} & 1 & \\ 2 & & 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3 \oplus 4.$$

It is easy to check that A is both left and right well-filtered, hence the standard filtrations of the indecomposable projective A-modules will be reflected by the composition series of the corresponding projective modules over A^+. On the other hand we have the composition series $0 \to S(2) \to S(2) \oplus S(4) \to P_{A^+}(1)$ over A^+, and this cannot correspond to any standard filtration of $P_A(1)$, because $P_A(1)$ does not have any submodules isomorphic to $\Delta_A(2)$. Note also that here the module $P_{A^+}(1)/S(2)$ is not the image of any module $M \in \mathscr{F}(\Delta)$ under the action of F^+.

The preceding results can be strengthened if there exists an (algebra) section map $s: A^+ \to A$, i.e. a map s such that its composition with the natural epimorphism $p: A \to A^+$ gives the identity map of A^+. In this situation A^+ can be canonically identified with a subalgebra of A.

The previous example (Example 2.7) shows that a right well-filtered quasi-hereditary algebra need not have a section map corresponding to A^+. On the other hand, let us just list some cases when A is quasi-hereditary well-filtered and it has a section map:

- If A is lean quasi-hereditary and monomial, then it is well-filtered by Proposition 1.10, and it is clear that the monomiality of the algebra gives a section map from A^+ to A.
- If A is the Ext-algebra of a lean quasi-hereditary monomial algebra then from Proposition 1.16 we get that A is well-filtered (and quasi-hereditary). Furthermore, the proof of Proposition 1.13 implies that A has a section map.
- If A is a shallow quasi-hereditary algebra, isomorphic to $K\Gamma/I$ for some graph Γ and set of relations I, then A will be well-filtered according to Proposition 1.9, and it will clearly have a section map.

Another sufficient condition for the existence of a section map is given for path algebras modulo relations by the following proposition.

Proposition 2.8. *Let $A = K\Gamma/I$, and assume that A^+ is hereditary. Then there exists a section map $s: A^+ \to A$.*

Proof. Let Γ' denote the graph which can be obtained from Γ by deleting the arrows $j \to i$ for $j > i$. Let the elements p_i be paths in Γ and $\alpha_i \in K$. Consider the element $r = \sum_i \alpha_i p_i \in K\Gamma$. Denote by r' the following element in $K\Gamma'$: $r' = \sum_i \alpha_i p_i'$, where $p_i' = p_i$ if p_i is a path in Γ' and 0 otherwise. Let $I' = \{ r' \in K\Gamma' \mid r \in I \}$. Then, clearly, $A^+ \simeq K\Gamma'/I'$ with Γ' being the graph of A^+. From the fact that A^+ is hereditary, we get that $I' = 0$. This means that every path summand of each element of I contains an arrow $j \to i$ with $j > i$. But then the map $K\Gamma \to K\Gamma/I$ maps $K\Gamma'$ isomorphically onto a subalgebra of $A = K\Gamma/I$, which maps isomorphically onto A^+ via the natural epimorphism. Hence we get a section map $s: A^+ \simeq K\tilde{\Gamma} \to A = K\Gamma/I$, as required. □

The existence of a section map $s: A^+ \to A$ relates closely to the concept of a Borel subalgebra of a quasi-hereditary algebra, as defined by König [14]. Let us first recall this concept.

Let K be an algebraically closed field. A subalgebra B of a basic quasi-hereditary algebra $(A,\boldsymbol{e})$ over the field K is called a *strong exact Borel subalgebra*, if B contains a maximal semisimple subalgebra which is also a maximal semisimple subalgebra of A (hence we can identify the simple A- and simple B-modules), and furthermore:

(i) B is directed (with respect to the order inherited from $(A,\boldsymbol{e})$) with simple standard B-modules;

(ii) A is projective as a left B-module (and hence the functor $-\otimes_B A$: mod-$B \to$ mod-A is exact);

(iii) for every index i there is an isomorphism $S_B(i)\otimes_B A \simeq \Delta_A(i)$.

Thus, strong exact Borel subalgebras describe the standard filtration of the projective A modules in a similar fashion as A^+ does when A is right well-filtered. Notice, however that in case of Borel-subalgebras, the connection between the composition structure of B and the standard filtration of A is given by the induction functor $G = -\otimes_B A$: mod-$B \to$ mod-A instead of the pushdown functor F^+ : mod-$A \to$ mod-A^+.

We have the following statement about the relationship of A^+ and Borel subalgebras.

Theorem 2.9. *Let K be algebraically closed and let $(A,\boldsymbol{e})$ be a quasi-hereditary K-algebra which is right well-filtered. Assume that there is a section map $s : A^+ \to A$. Then $s(A^+) = B$ is a strong exact Borel subalgebra of A.*

Proof. Since the ideal I^+ is entirely in the radical of A, the subalgebra B will clearly contain a maximal semisimple subalgebra which is a maximal semisimple subalgebra of A as well. Furthermore, it is equally clear that $B \simeq A^+$ is directed, with simple standard A^+-modules.

The fact that the module ${}_BA$ is projective, follows from Proposition 1.5(iv) and the quasi-heredity of A. Namely, the module ${}_AA$ has a filtration with factors isomorphic (as A-modules, hence also as B-modules) to some $\Delta^\circ(i)$. However, $\Delta^\circ(i)$ is a projective left A^+-module, hence ${}_BA$ must be projective.

Finally, we show that the simple A^+-modules induce the standard A-modules. It is easy to see that $F^+G(M) \simeq M$ for any $M \in$ mod-A^+. Indeed, $F^+G(M) = M\otimes_B A\otimes_A A^+$, and $A\otimes_A A^+$ is isomorphic to A^+ as an A^+-module, so F^+G is equivalent to the identity functor $1_{\text{mod-}A^+}$. Thus $F^+G(S(i)) \simeq S(i)$. Hence $G(S(i))$ has a simple top and must be a homomorphic image of $P_A(i)$. Note also that the right leanness of A (cf. Corollary 1.7) implies that $V_A(i)$ must be in the kernel of the epimorphism $P(i) \to G(S(i))$, hence $G(S(i))$ is an epimorphic image of $\Delta_A(i)$. Now, due to the exactness of G, a composition series of A^+ is mapped by G into a filtration of $G(A^+) \simeq A_A$, with factors equal to the induced modules $G(S(i))$. Hence the composition length of A_A satisfies:

$$\ell(A_A) \leq \sum_{i,j} [A^+_{A^+} : S_{A^+}(i)] \cdot [\Delta_A(i) : S_A(j)],$$

with equality holding if and only if each induced module $G(S(i)) \simeq \Delta_A(i)$. But the exactness of F^+ on $\mathscr{F}(\Delta)$ (cf. Theorem 2.3) implies that $[A^+_{A^+} : S_{A^+}(i)] = [A : \Delta(i)]$. Hence the right hand side is indeed $\ell(A_A)$. □

Let us mention here that the (right and left well-filtered) algebra given in Example 2.7 has no Borel subalgebras (cf. [14]). We should, however, mention, that there exist algebras which are not well-filtered but have Borel subalgebras. In fact, the algebra given in Example 1.8 illustrates this feature: it is not right well-filtered, but it is easy to check that it does have a strong exact Borel subalgebra.

Finally let us mention that in proving Theorem 2.9 we have also obtained the following result.

Proposition 2.10. *Let A be a quasi-hereditary algebra which is right well-filtered and which has a section map $A^+ \to A$. Then the functor $G = - \otimes_{A^+} A : \text{mod-}A^+ \to \text{mod-}A$ is an exact embedding of* mod-A^+ *into $\mathcal{F}(\Delta)$, the category of right A-modules having a standard filtration, while the restriction of the functor $F^+ : \text{mod-}A \to \text{mod-}A^+$ to the subcategory $\mathcal{F}(\Delta)$ is dense and full.*

Proof. The statement follows from the observation that $F^+G(M) \simeq M$ for every $M \in \text{mod-}A^+$. □

3. Special constructions

In this section we show that quasi-hereditary algebras arising in two known constructions satisfy the well-filtered property.

Let us first recall a construction of Auslander in [5]. Let R be an arbitrary finite dimensional K-algebra and let t be the nilpotency index of $\operatorname{rad} R$, i.e. assume that $\operatorname{rad}^{t-1} R \neq 0$ and $\operatorname{rad}^t R = 0$. Let us define the left R-module

$$ {}_R X = \bigoplus_{i=1}^{n} X_i, $$

where the modules X_i are all mutually non-isomorphic indecomposable (local) direct summands of $\bigoplus_{s=1}^{t} R/\operatorname{rad}^s R$, ordered in such a way that $i<j$ implies $Ll(X_i) \geq Ll(X_j)$. (Here $Ll(M)$ denotes the Loewy length of the module M.) Finally, let us define $A = \operatorname{End}_R X$. It was shown in [10] that A is quasi-hereditary with respect to the order inherited from the summands X_i. Our next result shows that A is right well-filtered (but not necessarily left well-filtered).

Theorem 3.1. *Let $A = \operatorname{End}_R(\oplus X_i)$ be the finite dimensional K-algebra as defined above. Then, with respect to the induced order of the simple A-modules, A is right well-filtered, and A^+ is a serial hereditary algebra. If K is algebraically closed, then there exists a section map $s : A^+ \to A$, and $s(A^+)$ is a strong exact Borel subalgebra of A.*

Proof. Let us denote by e_i the idempotent element of A, corresponding to the summand X_i. Thus the subspaces e_iAe_j can be identified with $\operatorname{Hom}_R(X_i, X_j)$, for $1 \leq i,j \leq n$.

First we show that $I^+ = \{ f \in A \mid \operatorname{Im} f \subseteq \operatorname{rad} X \}$ which will be identified with $\operatorname{Hom}_R(X, \operatorname{rad} X)$. In other words, the elements of I^+ are precisely those endomorphisms

f whose components $e_i f e_j \in \mathrm{Hom}_R(X_i, X_j)$ are not epimorphisms, i.e. $e_i f e_j \in \mathrm{Hom}_R(X_i, \mathrm{rad}\, X_j)$ for $1 \le i,j \le n$. Let $g \in e_j A e_i$ for some $j > i$. Then $Ll(X_j) \le Ll(X_i)$, where in case of equality the top composition factors are different. Hence g is not an epimorphism, so $g \in \mathrm{Hom}_R(X_j, \mathrm{rad}\, X_i)$. Thus $\mathrm{Im}\, g \subseteq \mathrm{rad}\, X$. Since I^+ as an ideal is generated by the sets $e_j A e_i$ for $j > i$ and since $\mathrm{Hom}_R(X, \mathrm{rad}\, X) \lhd \mathrm{End}_R(X) = A$, we get that $I^+ \subseteq \mathrm{Hom}_R(X, \mathrm{rad}\, X)$.

To show the opposite inclusion, assume that none of the components of $f \in A$ is an epimorphism. Since $f = \sum_{i,j} e_i f e_j$, it is enough to show that each of the components belongs to I^+. Thus we may assume that $f \in e_i A e_j$, and clearly we may restrict to the case when $i \le j$. But then $\ell = Ll(\mathrm{Im}\, f) < Ll(X_j) \le Ll(X_i)$, hence f can be factored through $X_t = X_i / \mathrm{rad}^\ell X_i$. The condition on the Loewy length of X_t clearly implies that $j < t$, hence $f \in e_i A e_t A e_j \subseteq I^+$.

We shall now prove that the condition (ii)$''$ from Proposition 1.4 holds for A. Consider an element $f \in e_j A e_i A e_k$ for some $i < j \le k$. Clearly, $f \in I^+$. Thus the considerations above show that $f \in \mathrm{Hom}_R(X_j, X_k)$ can be factored through X_t, where $X_t = X_j / \mathrm{rad}^\ell X_j$, with $\ell = Ll(\mathrm{Im}\, f) < Ll(X_k) \le Ll(X_j)$. Hence $t > k$, thus $f \in e_j A e_t k A e_k \subseteq e_j A \varepsilon_{k+1} A e_k$, as required.

Next, we show that A^+ is right serial and hereditary. To this end it is enough to show that $\mathrm{rad}\, e_i A^+$ is local and projective over A^+ for $1 \le i \le n$ (here $e_i \in A$ is identified with its natural image in A^+).

Let $\mathrm{Epi}_R(X_i, X_j)$ be the vector space $\mathrm{Hom}_R(X_i, X_j) / \mathrm{Hom}_R(X_i, \mathrm{rad}\, X_j)$. From the description of I^+ it is easy to see that $e_i A^+$ can be identified with $\bigoplus_j \mathrm{Epi}_R(X_i, X_j)$ (with the natural A^+ structure). It is also clear that $\mathrm{Epi}_R(X_i, X_j) \ne 0$ if and only if $X_j = X_i / \mathrm{rad}^t X_i$ for some $1 \le t \le Ll(X_i) = \ell$. This gives easily that $\mathrm{rad}\,(e_i A^+) = \bigoplus_{t=1}^{\ell-1} \mathrm{Epi}_R(X_i, X_i / \mathrm{rad}^t X_i)$, and in general, for $1 \le k \le \ell$, we have $\mathrm{rad}^k (e_i A^+) = \bigoplus_{t=1}^{\ell-k} \mathrm{Epi}_R(X_i, X_i / \mathrm{rad}^t X_i)$. Now, in general, if $Ll(X_j) < Ll(X_i) = \ell$, then $\mathrm{Hom}_R(X_i, X_j) \simeq \mathrm{Hom}_R(X_i / \mathrm{rad}^{\ell-1} X_i, X_j)$ (as right A-modules). Hence, if $X_i / \mathrm{rad}^{\ell-1} X_i = X_j$, then we get that $\mathrm{rad}\, e_i A^+ \simeq e_j A^+$, hence $\mathrm{rad}\, e_i A^+$ is local and projective for $1 \le i \le n$, as required.

As a consequence of our previous considerations, one can see that, given two epimorphisms $f_i : X_i \to X_k$ and $f_j : X_j \to X_k$, with $i < j < k$, f_i always factors through f_j. This just means that the algebra A^+ is left serial as well. (In other words, A^+ is a product of K-algebras whose quivers are directed paths.)

Finally, the statement about the existence of a section map $s : A^+ \to A$ follows from the heredity of A^+ and Proposition 2.8, while Theorem 2.9 implies that $s(A^+)$ is a strong exact Borels subalgebra of A. □

The following example illustrates that A is indeed not necessarily left well-filtered.

Example 3.2. Let

$$ {}_R R = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 2 & & 1 \\ 1 & & \end{smallmatrix} $$

be the (left) regular representation of the (hereditary) path algebra R over a field K. Take

$$ {}_RX = \begin{smallmatrix} & & 3 & \\ & 2 & & 1 \\ 1 & & & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 2 & & 1 \end{smallmatrix} \oplus 1 \oplus 2 \oplus 3 $$

and consider $A = \mathrm{End}_R(X)$. Then the (right well-filtered) algebra A has the following regular representation:

$$ A_A = \begin{smallmatrix} 1 \\ 3 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} & & 4 & & \\ & 2 & & 1 & \\ 1 & & & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 3 \end{smallmatrix} \oplus 6\,. $$

Furthermore,

$$ A^+_{A^+} = \begin{smallmatrix} 1 \\ 3 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 6 \end{smallmatrix} \oplus 4 \oplus 5 \oplus 6\,. $$

On the other hand, A is not lean, hence it is not left well-filtered.

Let us remark that $(A^*)^+ \not\simeq (A^+)^*$. Indeed, $\dim_K (A^*)^+ = 10$, while $\dim_K (A^+)^* = 9$. Explicitly,

$$ {}_{(A^*)^+}(A^*)^+ = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 6 \end{smallmatrix} \oplus 5 \oplus 6\,. $$

and

$$ {}_{(A^+)^*}(A^+)^* = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 6 \end{smallmatrix} \oplus 4 \oplus 5 \oplus 6\,. $$

Finally, we recall a construction due to Dlab, Heath and Marko (cf. [8]).

Let R be a commutative self-injective local algebra, finite dimensional over a splitting field K. Let $\{X_i \mid 1 \le i \le n\}$ be a set of distinct local ideals of R, indexed in such a way that $X_i \supset X_j$ implies $i < j$. Note that here the containment $X_j \subseteq X_i$ is equivalent to the existence of an epimorphism $X_i \to X_j$. Assume that $X_1 = R$, furthermore that $n = \dim_K R$ and for each index i we have $\mathrm{rad}\, X_i = \sum_j X_j$, where the summation is taken for those ideals X_j which are properly contained in X_i. Finally, let $A = \mathrm{End}_R(X)$, where $X = \bigoplus_{i=1}^n X_i$. The main result of [8] is that A is quasi-hereditary with respect to the inherited order of the summands of X and A admits a duality which keeps the simple modules $S(i)$ fixed.

Then we can prove the following.

Theorem 3.3. *Let A be the algebra of the DHM-construction, defined above. Then A is both left and right well-filtered. Moreover, there exists a section map $s : A^+ \to A$, and $s(A^+)$ is a strong exact Borel subalgebra of A.*

Proof. The existence of a duality implies that it is enough to show that A is right well-filtered. We are going to show that the condition (ii)$''$ of Proposition 1.4 holds for A. As in the proof of Theorem 3.1, denote by e_i the idempotent endomorphism corresponding to the summand X_i and let $f \in e_jAe_iAe_k$ for some $i < j \le k$. Then $f = f'f''$ with $f' \in e_jAe_i \simeq \mathrm{Hom}_R(X_j, X_i)$ and $f'' \in \mathrm{Hom}_R(X_i, X_k)$. Using an earlier remark, f' cannot be an epimorphism, otherwise we would get that $X_i \subseteq X_j$, implying $i > j$. Hence,

$f' \in \mathrm{Hom}_R(X_j, \mathrm{rad}\, X_i)$ and thus, $f \in \mathrm{Hom}_R(X_j, \mathrm{rad}\, X_k)$. Since $\mathrm{rad}\, X_k = \sum_\ell X_\ell$ for some indices $\ell > k$, Lemma 2 of [8] implies that f can be factored through the canonical map $\bigoplus_\ell X_\ell \to \sum_\ell X_\ell$. Thus $f \in e_j A \varepsilon_{k+1} A e_k$, as required. Note that – similarly to the situation of Theorem 3.1 – the previous argument also yields that $I^+ = \{ f \in A \mid \mathrm{Im}\, f \subseteq \mathrm{rad}\, X \}$.

To complete the proof, we have to show the existence of a section map $s : A^+ \to A$. Let us note first, that each (local) ideal X_i is isomorphic to the factor module $R/\mathrm{Ann}\, X_i$. Fixing such an isomorphism for every index i, let $x_i \in X_i$ be the coset of $1 \in R$ under this isomorphism. It is easy to see that if $X_i \supseteq X_j$ (hence $i < j$), then there is a unique epimorphism $f_{ij} : X_i \to X_j$, mapping x_i to x_j. Denote by B the K-subspace of A generated by the morphisms f_{ij} for every pair $X_i \supseteq X_j$. Since $f_{ij} f_{jk} = f_{ik}$ whenever $X_i \supseteq X_j \supseteq X_k$, B is a subalgebra. The explicit description of I^+, given above, implies that B is disjoint from I^+. Since K is a splitting field for R, the canonical epimorphism $A \to A^+$ maps B surjectively onto A^+. Hence, $B \simeq A^+$, giving the required section map. Theorem 2.9 implies that $s(A^+)$ is a strong exact Borel subalgebra of A. □

Observe that in the previous construction, A^+ can be described completely as follows. Let Γ be the graph with the set of vertices $\{1, 2, \ldots, n\}$, and put an arrow $i \to j$ if $X_i \supseteq X_j$ and no $k \neq i, j$ exists with $X_i \supseteq X_k \supseteq X_j$. Then $A^+ \simeq K\Gamma/I$ where the ideal I is generated by all relations $\alpha_{ij} - \beta_{ij}$, with α_{ij} and β_{ij} being two arbitrary paths between i and j.

Addendum

After completing their paper the authors have learnt that S. König in his paper "Cartan decompositions and BGG-resolutions" [*Manuscripta Math.* **86** (1995) 103–111], considered algebras having a Cartan decomposition for which every simple module has a BGG resolution. In particular he obtained the equivalence of Proposition 1.6(i) and (v) for this special situation.

References

[1] I. Ágoston, V. Dlab, E. Lukács, Lean quasi-hereditary algebras, in: Representations of Algebras, 6th Int. Conf., Ottawa, 1992, CMS Conf. Proc. 14 (1993) 1–14.
[2] I. Ágoston, V. Dlab, E. Lukács, Homological characterization of lean algebras, Manuscripta Math. 81 (1993) 141–147.
[3] I. Ágoston, V. Dlab, E. Lukács, Homological duality and quasi-heredity, Can. J. Math. 48 (1996) 897–917.
[4] I. Ágoston, V. Dlab, E. Lukács, Well-filtered algebras, Abstracts, Amer. Math. Soc. 908-16-164 (1996).
[5] M. Auslander, Representation dimension of Artin algebras, Queen Mary College Notes, London, 1971.
[6] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Differential operators on the base affine space and a study of g-modules, in: I.M. Gelfand (Ed.), Lie groups and their representations. Summer school of the Bolyai János Mathematical Society, Budapest, 1971, Akadémiai Kiadó, Budapest, 1975, pp. 21–64.
[7] E. Cline, B.J. Parshall, L.L. Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew. Math. 391 (1988) 85–99.

[8] V. Dlab, P. Heath, F. Marko, Quasi-hereditary endomorphism algebras, Can. Math. Bull. 38 (1995) 421–428.

[9] V. Dlab, C.M. Ringel, Quasi-hereditary algebras, Illinois J. Math. 35 (1989) 280–291.

[10] V. Dlab, C.M. Ringel, Every semiprimary ring is the endomorphism ring of a projective module over a quasi-hereditary ring, Proc. Amer. Math. Soc. 107 (1989) 1–5.

[11] V. Dlab, C.M. Ringel, The module theoretical approach to quasi-hereditary algebras, in: Representations of Algebras and Related Topics, London Math. Soc. Lecture Note Series 168, Cambridge University Press, Cambridge, 1992, pp. 200–224.

[12] V. Dlab, Appendix on quasi-hereditary algebras, in: Yu.A. Drozd, V.V. Kirichenko, Finite Dimensional Algebras, Springer, Berlin, 1994.

[13] E. Green, D. Zacharia, The cohomology ring of a monomial algebra, Manuscripta Math. 85 (1994) 11–23.

[14] S. König, A guide to exact Borel subalgebras of quasi-hereditary algebras, in: Representations of Algebras. 6th International Conf. Ottawa, 1992, CMS Conf. Proc. 14 (1993) 291–308.